$$xT^{1/3} = C. \tag{25}$$

The solution (25) corresponds to the steady state temperature distribution in a rod.

9. Let $f(T) = T^{-4/3}$. Then the operator $X_2 + X_5 = (1 + x^2)\,\partial/\partial x - 3xT\,\partial/\partial T$ generates the invariants τ and $(1 + x^2)T^{2/3}$ and the invariant solution has the form $(1 + x^2)T^{2/3} = v(\tau)$. The substitution $v(\tau)$ in (3) gives

$$v' = 2/v \text{ and } v^2 = 4\tau + C,$$

or

$$(1 + x^2)^2 T^{4/3} = 4\tau + C. \tag{26}$$

Finally we note that in view of the complexity of the calculations of $v(\mu)$ in the structure of the invariant solutions corresponding to some of the operators listed in (4) through (7), we do not give all of the results here. However, it was shown in examples 1-9 that the kinematic description of the process of unsteady heat conduction in terms of (3) can be used to determine the function $v(\mu)$, which can be a very difficult problem using the "traditional" approach based on equation (1).

Secondly, we note that the invariant solutions obtained here, which are related to the intermediate asymptotic solutions of [6-8], contain important information on the behavior of the general solutions of boundary-value problems for the nonlinear heat-conduction equation, both for the case of fixed boundaries (or moving boundaries with a known form of the motion) as well as for the case where the motion of the boundary in time is found from an additional condition (the Stefan condition) in the case of a phase transition of the material.

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FINITE-DIFFERENCE METHOD FOR SOLVING A ONE-DIMENSIONAL NONSTATIONARY PROBLEM OF RADIATIVE—CONDUCTIVE HEAT TRANSFER

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UDC 536.33

An algorithm and examples of the solution of problems of complex heat transfer are given.

In this work, an effective method is offered for solving one-dimensional nonstationary boundary-value problems of radiative—conductive heat transfer with the exact equations for radiative transfer [1]. In this paper, results are presented on further development of works [2, 3], and examples of calculations and comparisons with known results of other authors are given.

We consider a flat layer of an emitting, absorbing, and anisotropically scattering medium with optically smooth or diffusely reflecting partially transparent surfaces. Initially, a nonuniformly heated layer is placed in the medium, the temperature and the coefficient of heat emission of which change according to a given law. Under the condition of azimuthal symmetry, we write the equations of radiative transfer in the form [1]

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Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 51, No. 5, pp. 840-847, November, 1986. Original article submitted August 12, 1985.

$$\mu \frac{\partial I^+}{\partial x} = -\alpha I^+ + \frac{\beta}{2} \int_0^1 [p(\mu', \mu) I^+(\mu') + q(\mu', \mu) I^-(\mu')] d\mu' + b,$$

$$-\mu \frac{\partial I^-}{\partial x} = \alpha I^- + \frac{\beta}{2} \int_0^1 [q(\mu', \mu) I^+(\mu') + p(\mu', \mu) I^-(\mu')] d\mu' + b. \tag{1}$$

Equations (1) are stationary, and time enters them as a parameter. The temperature dependence of the thermal radiation properties of the material is assumed to be given for each wavelength of the radiation.

For optically smooth surfaces $x = 0$ and $x = \ell$ the boundary conditions are of the form

$$I^+(0, \mu) = R_2 I^-(0, \mu) + n^2(1 - R_1) I_e^+(\mu'),$$
$$I^-(l, \mu) = R_2 I^+(l, \mu) + n^2(1 - R_1) I_e^-(\mu'). \tag{2}$$

The coefficient of reflection R_2 is calculated from the Fresnel formulas, where $R_2 = 1$ for all angles exceeding Brewster's angle.

For diffuse reflection at the surfaces, the conditions at the boundaries are

$$I^+(0, \mu) = 2r_2 \int_0^1 I^-(0, \mu') \mu' d\mu' + I_e^+(\mu'),$$
$$I^-(l, \mu) = 2r_2 \int_0^1 I^+(l, \mu') \mu' d\mu' + I_e^-(\mu'). \tag{3}$$

The equations of heat conduction for a plane layer are

$$c\rho \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(K(x, t) \frac{\partial T}{\partial x}\right) + 2\pi \int_{(\lambda_{\text{lim}})} d\lambda \int_0^1 \varkappa [(I^+ + I^-) - 2n^2 B] d\mu d\lambda. \tag{4}$$

Boundary conditions for Eq. (4) are

$$-K l_{11} \frac{\partial T}{\partial x} = 2\pi l_{12} \int_{(\lambda_n)} d\lambda \int_0^1 \varepsilon^+ (I_e^+ - B^+) \mu d\mu + l_{13} h_e^+ (T_e - T),$$

$$K l_{21} \frac{\partial T}{\partial x} = 2\pi l_{22} \int_{(\lambda_n)} d\lambda \int_0^1 \varepsilon^- (I_e^- - B^-) \mu d\mu + l_{23} h_e^- (T_e - T). \tag{5}$$

Indices + and − refer to the surfaces $x = 0$ and $x = \ell$, respectively. The coefficients ℓ_{ij} take on the values 0 or 1 and form a matrix of the boundary conditions, which allow us to take into account different versions of heat transfer on the surfaces of a layer.

Thus, mathematically speaking, solution of the generalized boundary-value problem of radiative–conductive heat transfer is reduced to solution of a coupled system of two integrodifferential equations (1), (4), with given boundary conditions (2), (5) or (3), (5). In order to find solutions, we use the method of finite differences. We introduce the design grid, and as nodes we select the points with coordinates

$$x_i = (i-1)\Delta x \, (i = 1, 2, \ldots, L); \quad \mu_m = (m-1)\Delta\mu \, (m = 1, 2, \ldots, M);$$
$$\lambda_j = (j-1)\Delta\lambda \, (j = 1, 2, \ldots, j); \quad t_n = (n-1)\Delta t \, (n = 1, 2, \ldots, N).$$

We designate the desired functions at the nodes of the grid

$$I^{\pm}_{i,m,n} = I^{\pm}(x_i, \mu_m, \lambda_j, t_n); \quad T_i^n = T(x_i, t_n).$$

At the initial moment of time $t = 0$ ($n = 1$), we find the field of the intensities of radiation $I^{\pm}_{i,m,n}$ for all the values λ_j from the given distribution of temperatures $T_{(x)}^{0}$ by using a numerical method of solving the first boundary-value problem [Eqs. (1), boundary conditions (2) or (3)]. For solving the coupled system of equations (1), (4) for the following moment of time $t = \Delta t$, we use an iterative approach. In order to find the temperature field, we use the solution $I^{\pm}_{i,m,n}$, related to the preceding moment of time, as a zero approximation to the functions $I^{\pm}_{i,m,n+1}$, entering the integral term of Eq. (4). From the determined temperatures T_i^{n+1}, we find the first approximation to $I^{\pm}_{i,m,n+1}$. Then we repeat the iteration until the given condition of convergence of the computational process for temper-

TABLE 1. Spatial-Angular Distribution of Intensities of Radiation at $b = 1$, $n = 1$, $I_e^+ = I_e^- = 0$, $\omega = 0$, $\alpha\ell = 1$

No.	μ	$x=0$ I^-	$x=l/2$ I^-	$x=l/2$ I^+	$x=l$ I^+	No.	μ	$x=0$ I^-	$x=l/2$ I^-	$x=l/2$ I^+	$x=l$ I^+
1	1,0000	0,630	0,392	0,392	0,630	2	1,0000	0,632	0,392	0,392	0,632
	0,9808	0,637	0,398	0,398	0,637		0,9659	0,644	0,404	0,404	0,644
	0,9239	0,659	0,416	0,416	0,659		0,8660	0,684	0,438	0,438	0,684
	0,8315	0,697	0,450	0,450	0,697		0,7071	0,756	0,506	0,506	0,756
	0,7071	0,754	0,504	0,504	0,754		0,5000	0,864	0,632	0,632	0,864
	0,5556	0,832	0,590	0,590	0,832		0,2588	0,979	0,855	0,855	0,979
	0,3827	0,924	0,725	0,725	0,924		0,0000	1,000	1,000	1,000	1,000
	0,1951	0,993	0,918	0,918	0,993						
	0,0000	1,000	1,000	1,000	1,000						

Note. Results of the authors, part No. 1; from work [8], part No. 2.

atures and intensities of radiation is not satisfied. The described sequence of operations is repeated for each subsequent moment of time.

For solving the problem of radiative heat transfer at the moment of time t_n, we replace the integrals by finite sums for each wavelength λ_i using one of the quadrature formulas, and we write Eqs. (1) for all the node points μ_m of the angular coordinate:

$$\mu_m \frac{\partial I_m^+}{\partial x} = -\alpha I_m^+ + \frac{\beta}{2} \sum_{k=1}^{M} (p_{mk} I_k^+ + q_{mk} I_k^-) h_k \sin\theta_k + b,$$
$$\mu_m \frac{\partial I_m^-}{\partial x} = \alpha I_m^- - \frac{\beta}{2} \sum_{k=1}^{M} (q_{mk} I_k^+ + p_{mk} I_k^-) h_k \sin\theta_k - b. \tag{6}$$

In the given calculations, the Simpson formula was used. It was assumed that M = 9, and L = 101.

We introduce M-dimensional column-vectors ϕ, Ψ with the components I_m^+, I_m^- and ϕ', Ψ', the components of which are the corresponding derivatives of the intensities. We write the obtained differential boundary-value problem (6), (2) or (6), (3) in the vector-matrix form

$$C\varphi' = -\alpha\varphi + P\varphi + Q\Psi + B, \quad C\Psi' = \alpha\Psi - Q\varphi - P\Psi - B, \tag{7}$$

where C, P, and Q are square matrices of coefficients μ, p, q; and B is the vector with components b.

The boundary conditions (2), (3) are now written in the form

$$\varphi(0) = R\Psi(0) + g_1, \quad \Psi(l) = R\varphi(l) + g_2. \tag{8}$$

In this case, the boundary conditions (2) in the form (8) correspond to the diagonal matrix R, the diagonal elements of which are the coefficients of the internal reflection from the surface for the given values of the angles. The boundary conditions (3) correspond to the square matrix R, obtained by multiplying the column with the elements r_2 by the row of the quadrature coefficients with the weighting coefficient μ.

In order to solve the obtained differential boundary-value problem (7), (8), we apply the method of finite differences

$$\varphi_1 = R\Psi_1 + g_1,\ C(\varphi_{i+1} - \varphi_i) = -\alpha\Delta x\varphi_{i+1} + \Delta x P\varphi_{i+1} + \Delta x Q\Psi_{i+1} +$$
$$+ \Delta x B_{i+1},\ C(\Psi_{i+1} - \Psi_i) = \alpha\Delta x\Psi_i - \Delta x Q\varphi_i - \Delta x P\Psi_i - \Delta x B_i, \tag{9}$$
$$\Psi_L = R\varphi_L + g_2, \quad i = 1, 2, \ldots, L-1.$$

The inner equations of system (9) are written on a two-point template, the first of them being related to the right point, and the second, to the left point, which provides for compatibility between the differential equations and the boundary conditions, and the solvability of the system as a whole.

Allowing for the form of the first boundary condition, we reduce system (9) (without the second boundary condition) to the form

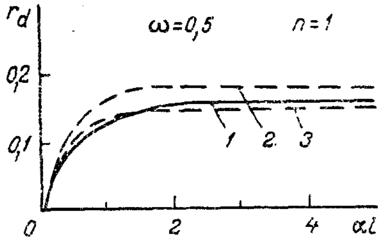

Fig. 1

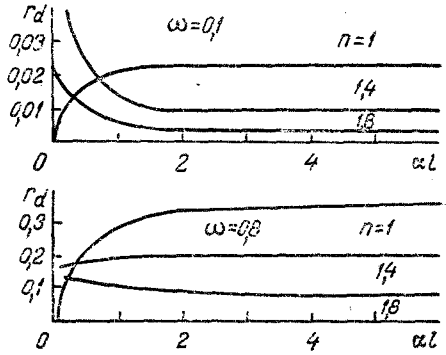

Fig. 2

Fig. 1. The dependence of r_d on $\alpha\ell$ for isotropic scattering with one-sided irradiation of a nonemitting layer: 1) numerical solution of the exact equations for radiation transfer; 2) two-flow solution; 3) modified method of mean flows.

Fig. 2. The dependence of r_d on $\alpha\ell$ for isotropic scattering with one-sided irradiation of a nonemitting layer with optically smooth partially transparent surfaces, I_e = const.

$$\varphi_i = G_i\Psi_i + v_i, \quad \Psi_i = H_i\Psi_{i+1} + w_i, \tag{10}$$

where G_i and H_i are square (M × M) matrices, v_i, w_i, are M-dimensional vectors, for the determination of which the following recursive relations are found:

$$\begin{gathered} H_i = (S - \Delta x Q G_i)^{-1} C, \quad w_i = (S - \Delta x Q G_i)^{-1} \Delta x (B_i + Q v_i), \\ G_{i+1} = S^{-1}(C G_i H_i + \Delta x Q), \ v_{i+1} = S^{-1}(C G_i w_i + C v_i + \Delta x B_{i+1}), \\ S = \alpha \Delta x E + C - \Delta x P, \end{gathered} \tag{11}$$

where E is the unit matrix, $G_1 = R_2$, $v_1 = g_1$.

The first iteration of computations, the direct pivotal method, consists in determining the matrix coefficients of system (10) based on recursive equations (11) for i = 1, 2, ..., L − 1. The iteration of the reverse pivotal method begins from computing the vector Ψ_L from the second boundary condition

$$\Psi_L = (E - R_2 G_L)^{-1}(g_2 + R_2 v_L). \tag{12}$$

Then for i = L − 1, L − 2, ..., 1 from system (10), we determine all the vectors ϕ_i, Ψ_i. Ultimately, for each moment of time and for each wavelength, the distribution of intensities of radiation is found in the layer according to the linear and angular coordinates. From the determined distributions of intensities, any integral characteristics of the radiation are easily determined. In particular, the value of the density of the radiative heat flow, required for solving the second boundary-value problem, is computed.

In order to solve the boundary-value problem of heat conduction using the method of finite differences, the equations and boundary conditions of which (4), (5) contain functions depending on the temperature in a significantly nonlinear way, we linearize the given equations according to the Newton–Kantorovich method

$$B(\lambda, T^{n+1}) = B(\lambda, T^n) + \left.\frac{\partial B}{\partial T}\right|_{T=T^n} (T^{n+1} - T^n), \tag{13}$$

$$I^{\pm}(x, \mu, \lambda, t_{n+1}) = I^{\pm}(x, \mu, \lambda, t_n) + \left.\frac{\partial I^{\pm}}{\partial t}\right|_{t=t_n} \Delta t. \tag{14}$$

From the linearized system of equations, we determine the desired temperatures by using the pivotal method. At each moment of time, we refine iteratively the obtained values, as described above.

The proposed method for solving the problem of radiative–conductive heat transfer allows us to determine the field of intensities of radiation and the field of temperatures in the layer under conditions of unsteady heat transfer.

Most results for problems of the given type are obtained with the use of the simplified mathematical models [4, 5].

TABLE 2. The Dependence of r_d on $\alpha\ell$, ω and n for I_e = const

αl		0,05			3,2		
F		0,5	0,75		0,5	0,75	
n	ω	β*					
		0,33	0,39	0,43	0,33	0,39	0,43
Two-sided irradiation of a nonemitting layer, b = 0							
1,0	0,1	0,9202	0,9202	0,9202	0,0394	0,0340	0,0319
	0,5	0,9534	0,9534	0,9534	0,1835	0,1600	0,1489
	0,9	0,9902	0,9902	0,9902	0,6125	0,6000	0,5922
	1,0	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
1,4	0,1	0,3258	0,3256	0,3256	0,0171	0,0127	0,0117
	0,5	0,4004	0,3995	0,3992	0,0837	0,0596	0,0550
	0,9	0,5780	0,5764	0,5759	0,3735	0,3344	0,3293
	1,0	0,6688	0,6669	0,6664	0,8359	0,8033	0,7963
1,8	0,1	0,0804	0,0803	0,0803	0,0056	0,0039	0,0034
	0,5	0,1235	0,1231	0,1230	0,0289	0,0187	0,0161
	0,9	0,3312	0,3305	0,3302	0,1617	0,1380	0,1335
	1,0	0,6199	0,6194	0,6197	0,7141	0,6928	0,6879
One-sided irradiation of an emitting layer, b = κ							
1,0	0,1	0,0838	0,0826	0,0826	0,9831	0,9820	0,9818
	0,5	0,0672	0,0615	0,0614	0,9667	0,9558	0,9532
	0,9	0,0488	0,0384	0,0384	0,8592	0,8011	0,7903
	1,0	0,0439	0,0323	0,0323	0,7121	0,6256	0,6104
1,4	0,1	0,0749	0,0745	0,0744	0,2711	0,2700	0,2670
	0,5	0,1182	0,1162	0,1157	0,2711	0,2700	0,2670
	0,9	0,2215	0,2179	0,2170	0,4558	0,4050	0,3907
	1,0	0,2742	0,2702	0,2692	0,6249	0,5524	0,5382

A distinguishing characteristic of the given algorithm is the fact that it is based on the use of the exact equations for the radiative transfer; this allows one to use it for estimating the accuracy of the approximate methods. In addition, as distinct from the two-flow schemes, the result of the solution of the problem of radiative transfer is the distribution of intensities of radiation in a medium with random scattering. This enables one to find any integral characteristics of the radiation and to examine the influence of scattering on their magnitudes.

In the problem of radiative–conductive heat transfer, the basic difficulty is in solving the first boundary-value problem (1), (2) or (1), (3), while the methods for solving the problem of heat conductivity are known, therefore, we will illustrate the potentialities of the above-described method with examples of investigation of radiative transfer.

In work [6], an analytic solution was obtained for a number of problems of radiative transfer in a half-space of an isotropically scattering medium by the method of Case [7], and this solution was compared with the numerical one. For example, for one-sided irradiation of the nonemitting half-space at n = 1 from the diffusely radiating source, the following expression was obtained for the dimensionless spatial density of the outgoing monochromatic radiation:

$$P^-(0) = \frac{2I_e}{\omega}(\sqrt{1-\omega}+\omega-1).$$

It is believed that the integral characteristics for the radiation of a layer and of the half-space differ only slightly at $\alpha\ell > 3$ for all the values of ω and n = 1.

In work [8], the angular distribution of the intensities of the radiation is found in the layer of an emitting, absorbing, and isotropically scattering medium when n = 1 and there is no external radiation. The conducted comparison (see Table 1) confirms the agreement of the results obtained.

For one-sided irradiation by a diffusely emitting source I_e = const of a nonemitting layer $b = 0$, $n = 1$, a comparison of the results is conducted for calculating the hemispherical reflectivity, obtained by a numerical solution of the exact equations of the radiative transfer, with the solution, found with the use of the two-flow approximation [6], and with the approximate solution obtained by the modified method of the mean flows, obtained by the authors of works [9, 10]. This comparison allows one to judge the accuracy of the approximation methods mentioned above (Fig. 1).

In Fig. 2, the relation is shown between r_d and $\alpha\ell$ at $n > 1$ (analogous results are not encountered in the literature). From an analysis of the graphs (see Fig. 2), it follows that the integral radiation characteristics of a layer tend to the corresponding characteristics of the half-space at $\alpha\ell > 3$ for all values of ω and $n \geq 1$. This conclusion holds also for anisotropic scattering (Table 2).

We consider the influence of anisotropy of scattering on the hemispherical reflectivity of a layer

$$r_d = \frac{2\pi \int\limits_0^1 \frac{1}{n^2} (1 - R_2) I^-(0, \mu) \mu d\mu}{2\pi \int\limits_0^1 (1 - R_1) I_e^+(\mu') \mu' d\mu'}.$$

In order to allow for anisotropy of scattering, the indicatrix of the spatial scattering was expanded in a series in Legendre polynomials with three terms being retained. Two characteristics of the indicatrix of scattering were introduced: F, the fraction of radiation scattered forward by particles, and β^*, the smoothness:

$$F = \frac{1}{2} \int\limits_0^1 \gamma(\mu_0) d\mu_0; \quad \beta^* = \frac{1}{2F} \int\limits_0^1 \gamma(\mu_0) \mu_0^2 d\mu_0.$$

Three types of the indicatrix are considered, isotropic scattering ($F = 0.5$, $\beta^* = 0.33$), linearly anisotropic ($F = 0.75$, $\beta^* = 0.39$), maximum extension forward of the indicatrix ($F = 0.75$, $\beta^* = 0.43$).

In Table 2, results of calculating r_d of a layer of the anisotropically scattering medium are given for two-sided external irradiation of a nonemitting layer and for one-sided irradiation with allowance for the intrinsic radiation of the material.

In work [11], it was assumed that r_d of a layer depends slightly on β^* and is determined basically by the parameter F; this is confirmed by calculations given in [9] and in our calculations for $n = 1$ for one-sided external irradiation of a nonemitting layer. The assumption that r_d of a layer depends only on the fraction F of radiation scattered forward can be extended to cover media having coefficients of refraction different from unity. However, from an analysis of the data from Table 2 it follows that the inaccuracy of such an approach increases with increase in the refractive index and optical depth.

At small $\alpha\ell$, the influence of scattering anisotropy on the value is unimportant. Therefore, we can consider the scattering as isotropic and use simpler mathematical models for calculations.

The algorithm for solving the problem of radiative–conductive heat transfer in a similar exact treatment is given in [12]; however, the results of the solution are not given so it is not possible to perform a comparison.

NOTATION

T_e, ambient temperature; h_e, heat-transfer coefficient; I^+, I^-, intensities of the beams making acute angles with the internal normals to the surfaces $x = 0$ and $x = 1$; θ', θ, angles characterizing the direction of the incident and scattered beams; $\mu = \cos\theta$; $\mu' = \cos\theta$; I_e, intensity of external radiation; $\alpha = \kappa + \beta$, κ, β, spectral absorption and scattering coefficients; p, q, integrals over the azimuthal angle from the scattering indicatrix $\gamma(\mu_0)$, $\mu_0 = \cos\theta_0$, $\theta_0 = \theta' - \theta$; b, internal sources of spatial radiation; r_1, r_2, spectral coefficients of diffusive reflection from the outer and inner surfaces of the boundaries of the layer; R_1, R_2, coefficients of mirror reflection from the outer and inner surfaces of the boundaries of the layer; c, specific heat capacity; ρ, density; k, coefficient of thermal conductivity; n, relative refractive index of the medium; h_k, coefficients of the quadrature formula; ω, spectral albedo; ε, emissivity.

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